The discharging method

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the 6th KIAS Combinatorics Workshop

Outline

- About the discharging method
  - What and why?
  - Example
- Application
- Main theorem and proof
- Open problems

General Steps

Consider a counterexample G

Find some configurations that G can not have

Assign some charges to the vertices and faces

Move the charge around

Show the initial charge sum and the final charge sum are different

So there are no counterexamples

Example

Every planar graph with girth at least 4 is 4-colorable.

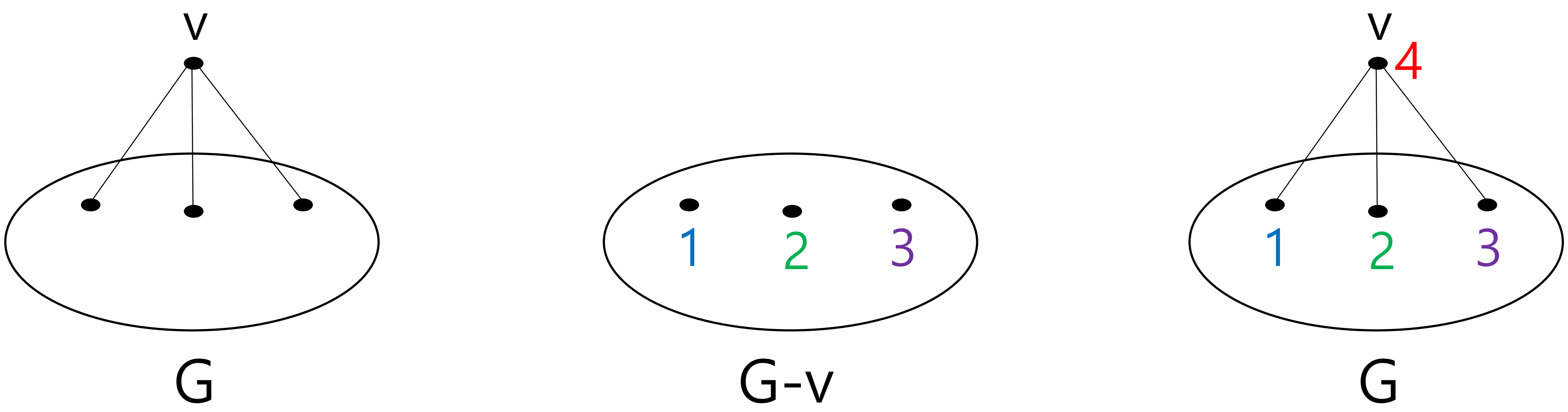
The girth of a graph is the length of a shortest cycle in the graph.

Example

Every planar graph with girth at least 4 is 4-colorable.

Let G be a minimal counterexample.

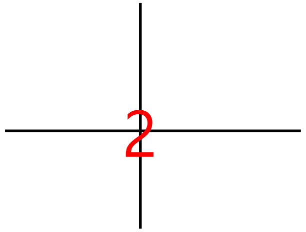
1. G has minimum degree at least 4.



Every planar graph with girth at least 4 is 4-colorable

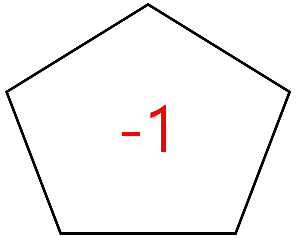
Charge of a vertex v

Assign 2d(v)-6 where d(v) is the degree of v



Charge of a face f

Assign d(f)-6 where d(f) is the length of f



Initial charge

degree	**2**	**3**	**4**	**5**	**6**	**7**	**8**
v=2d-6	-2	0	2	4	6	8	10
f=d-6	-4	-3	-2	-1	0	1	2

Every planar graph with girth at least 4 is 4-colorable

Initial charge

degree	2	3	4	5	6	7	8
v=2d-6	-2	0	2	4	6	8	10
f=d-6	-4	-3	-2	-1	0	1	2

Initial charge sum

=vertex charge + face charge

$=\sum_v (2d(v)-6)+\sum_f (d(f)-6)$

$=2\sum_v d(v)-6|V(G)|+\sum_f d(f)-6|F(G)|$

$=4|E(G)|-6|V(G)|+2|E(G)|-6|F(G)|$

$=-6|V(G)|+6|E(G)|-6|F(G)|=-12<0$ (∵ Euler's formula)

Every planar graph with girth at least 4 is 4-colorable

Initial charge

degree	2	3	4	5	6	7	8
v=2d-6	X	X	2	4	6	8	10
f=d-6	X	X	-2	-1	0	1	2

Every planar graph with girth at least 4 is 4-colorable

Initial charge

degree	2	3	4	5	6	7	8
v=2d-6	X	X	2	4	6	8	10
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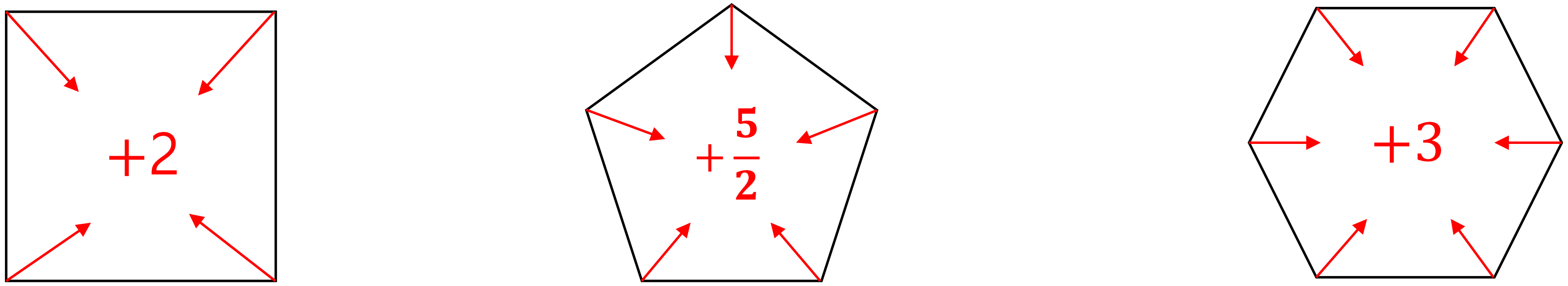
Discharging Rule

1. Each vertex sends charge $\frac{1}{2}$ to every incident face.



Every planar graph with girth at least 4 is 4-colorable

Initial charge

degree	2	3	4	5	6	7	8
v=2d-6	X	X	2	4	6	8	10
f=d-6	X	X	-2	-1	0	1	2

Rule1: Each vertex sends charge $\frac{1}{2}$ to every incident face.

Final charge

degree	2	3	4	5	6	7	8
v=2d-6	X	X	**0**	**1.5**	**3**	**4.5**	**6**
f=d-6	X	X	**0**	**1.5**	**3**	**4.5**	**6**

Final charge sum = nonnegative!!

Every planar graph with girth at least 4 is 4-colorable

Final charge

degree	2	3	4	5	6	7	8
v=2d-6	X	X	0	1.5	3	4.5	6
f=d-6	X	X	0	1.5	3	4.5	6

Final charge sum = nonnegative

Initial charge sum $\neq$ Final charge sum
Contradiction!!

There are no counterexamples.

Why?

Consider a counterexample G

Find some configurations that G can not have

there is no vertex of degree at most 3 (G is locally **dense**)

Assign some charges to the vertices and faces

in terms of degree

Move the charge around

Show the initial charge sum and the final charge sum are different

initial charge is negative since G is planar, which means **sparse**

So there is no counterexample

contradiction between **locally dense** and **globally sparse**

Applications : Coloring

Theorem(Appel and Haken, 1976)

Every planar graph is 4-colorable.

Theorem(Choi, Choi, J., and Suh, 2014+)

Every planar graph with girth at least 5 is (1,10)-colorable.

Applications : Decomposition

Theorem(Kim, Kostochka, West, Wu, and Zhu, 2013)

If a graph is sparse(formally, if it has maximum average degree less than $2+\frac{2d}{d+2}$),
then it decomposes into a forest and a graph of maximum degree at most d.

Applications : Long Cycle

Theorem(Kral, Pangrac, Sereni, and Skrekovski, 2009)

Let G be a fullerene graph with n vertices.

Then G contains a cycle of length at least $\frac{5}{6}n - \frac{2}{3}$.

Applications : Dominating Set

Theorem(Kowalik, 2012)

Let G be a graph with no isolated vertices such that every pair of degree 1 vertices is at distance at least 5 and every pair of degree 2 vertices is at distance at least 2.

Then G has a dominating set of size at most $\frac{3}{7}|V(G)|$.

Main theorem

Definition

A graph G is (x,y)-colorable if there exists
a partition V(G) into two parts satisfying that
one part has maximum degree at most x
another part has maximum degree at most y
e.g. 2-colorable = (0,0)-colorable

Theorem(Choi, Choi, J., and Suh, 2014+)

Every planar graph with girth at least 5 is (1,10)-colorable.

Main theorem

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Theorem(Choi, Choi, J., and Suh, 2014+)

Every planar graph with girth at least 5 is (1,16)-colorable.

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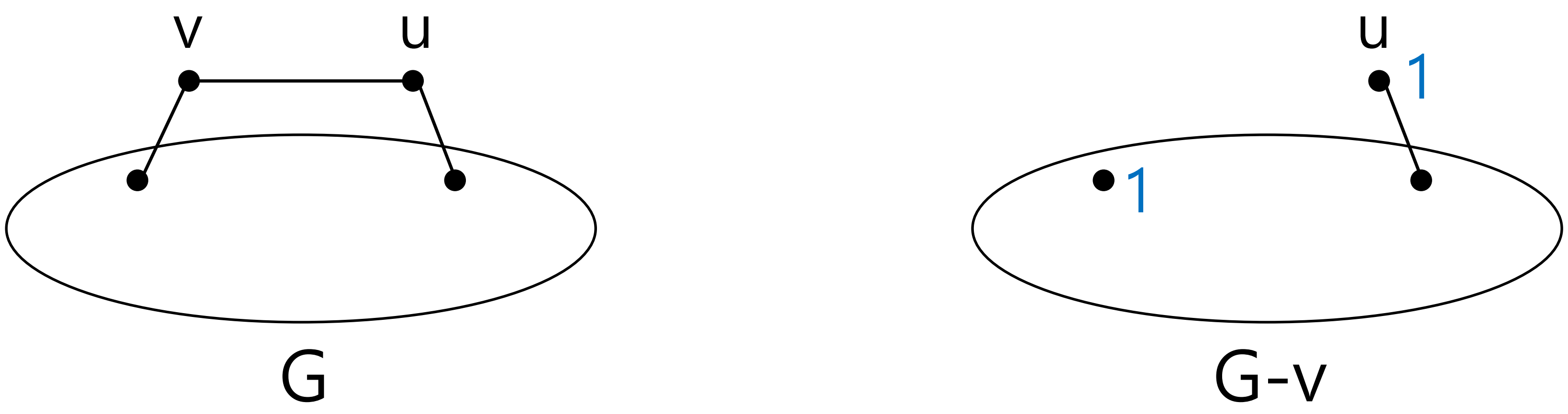
Let G be a minimal counterexample.

1. no 2-vertices adjacent to each other.



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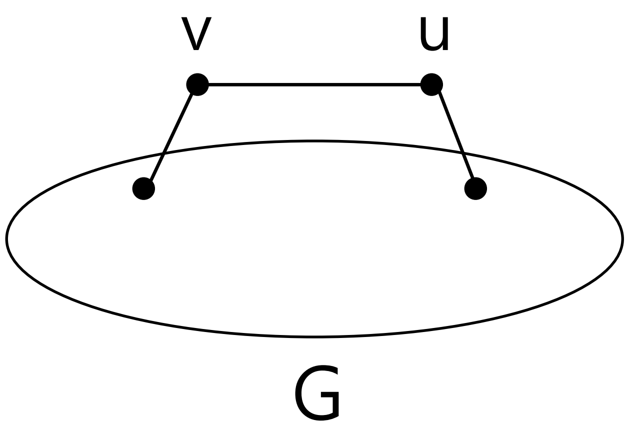

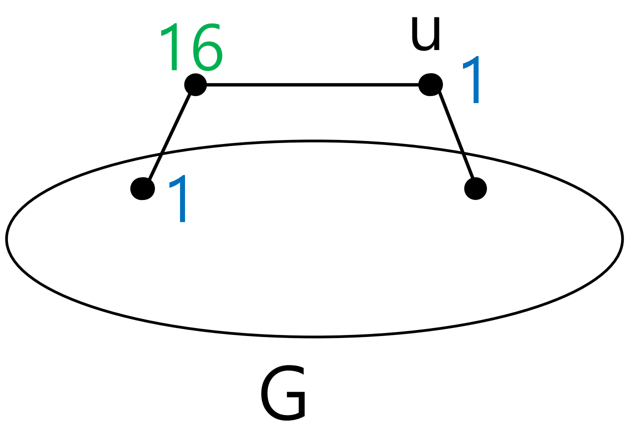

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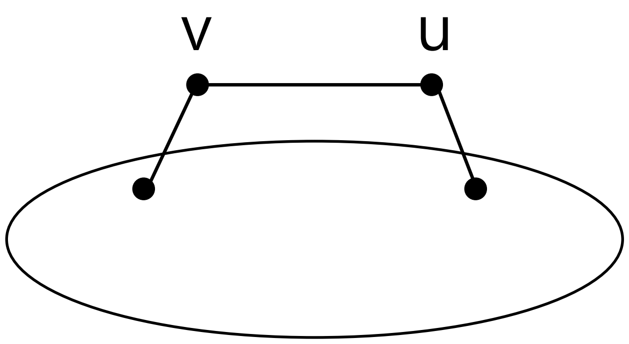
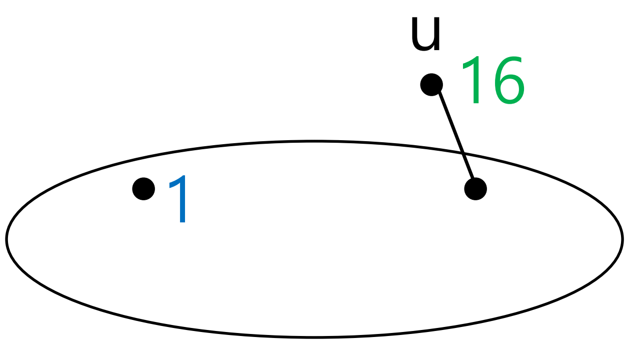


G



G-v

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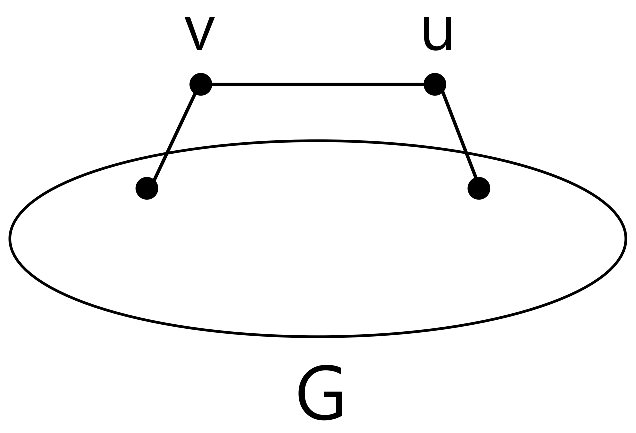

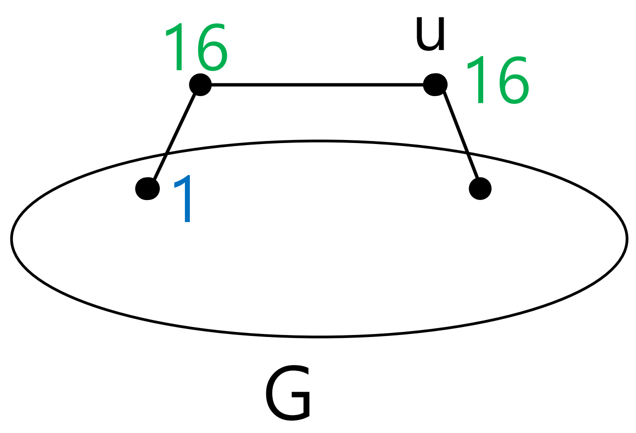

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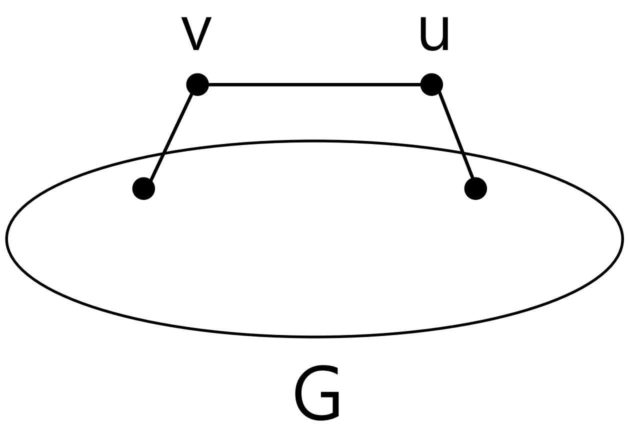

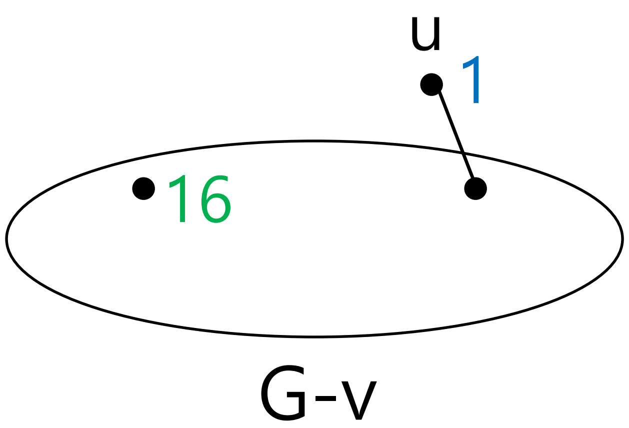

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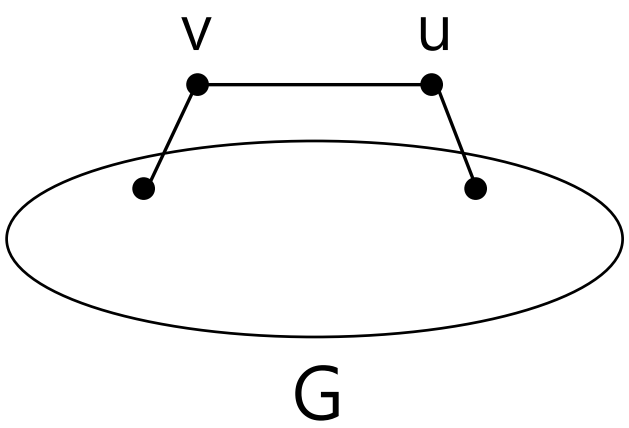

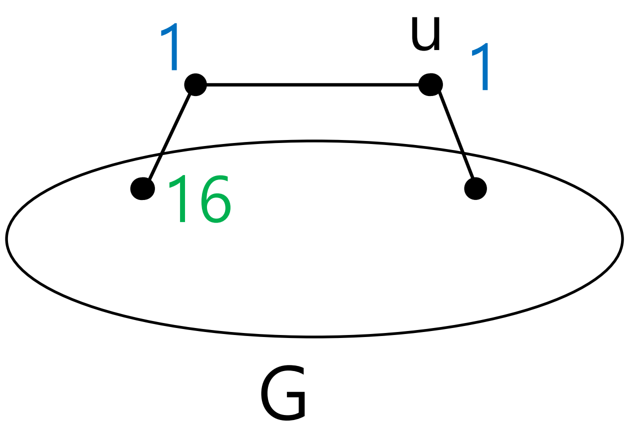

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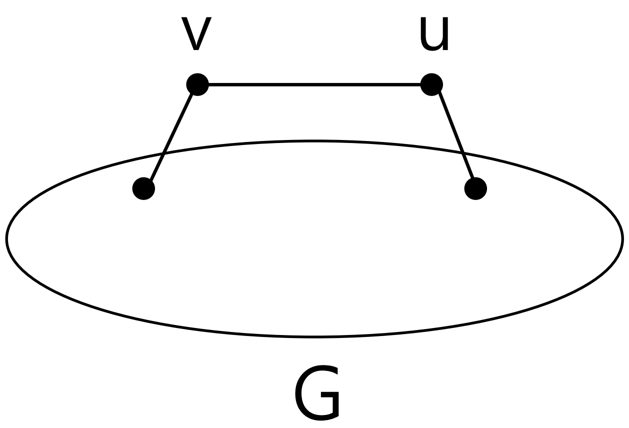

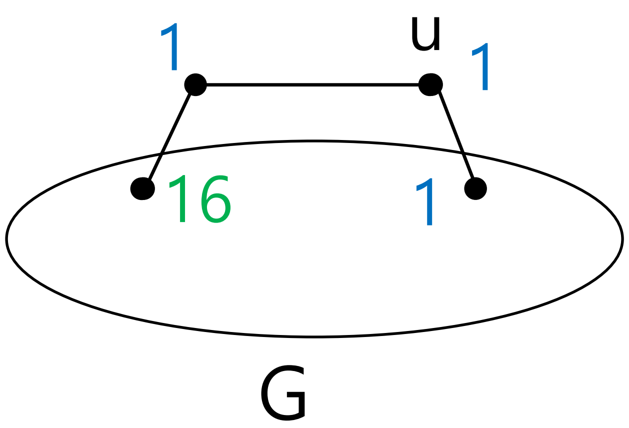

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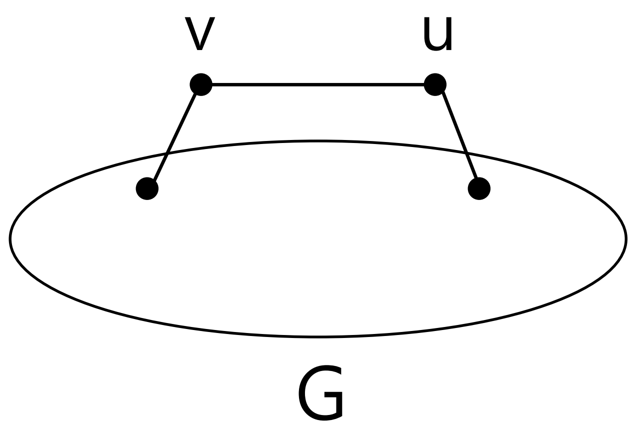
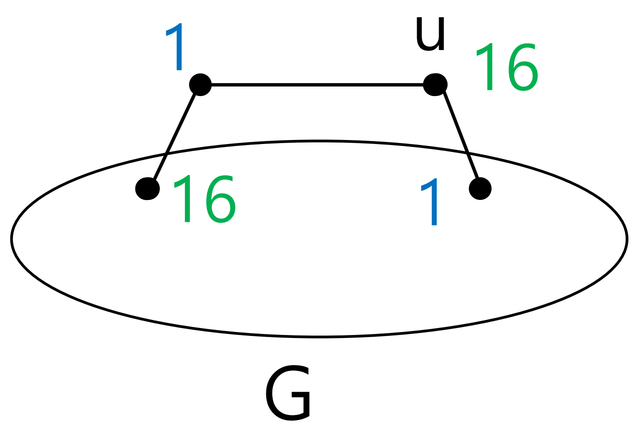
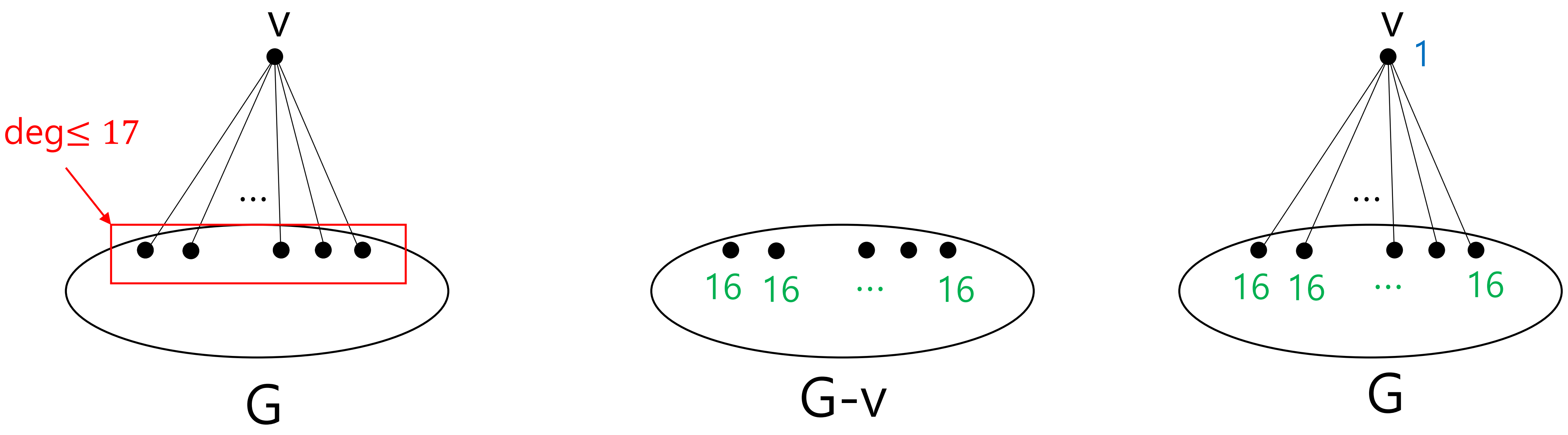




Every planar graph with girth at least 5 is (1,16)-colorable.

Let G be a minimal counterexample.

1. no 2-vertices adjacent to each other.
2. If v has degree at most 17, then v has a neighbor with degree at least 18.



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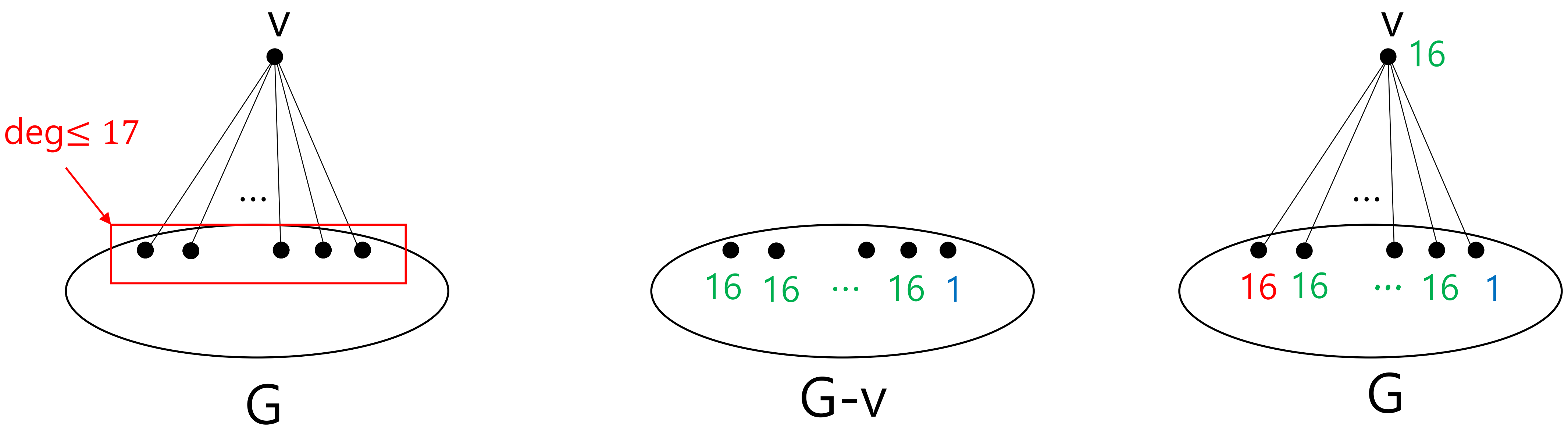

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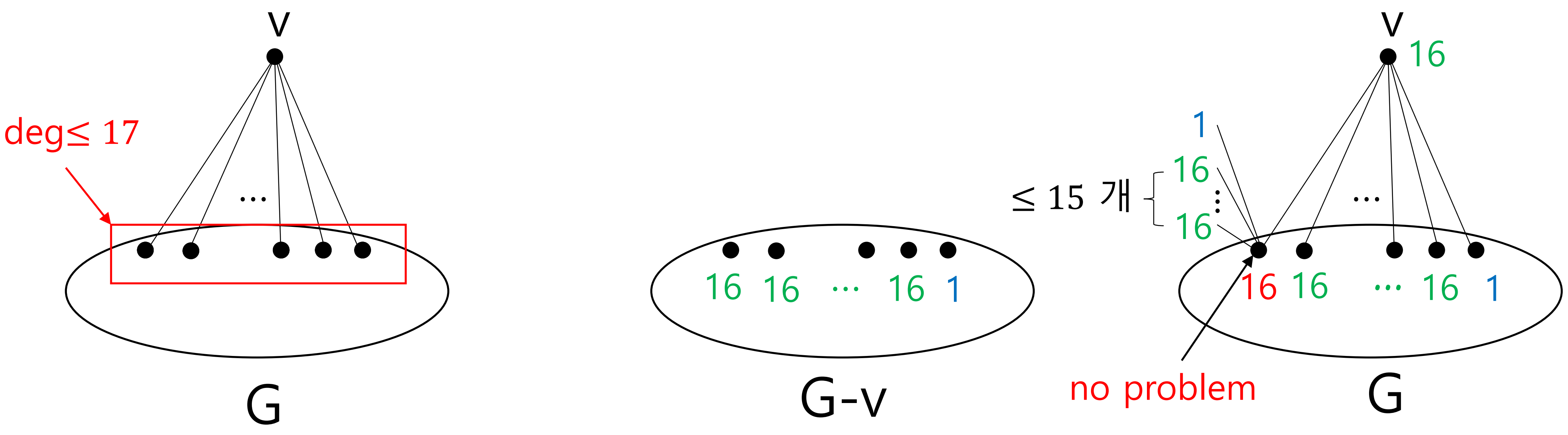

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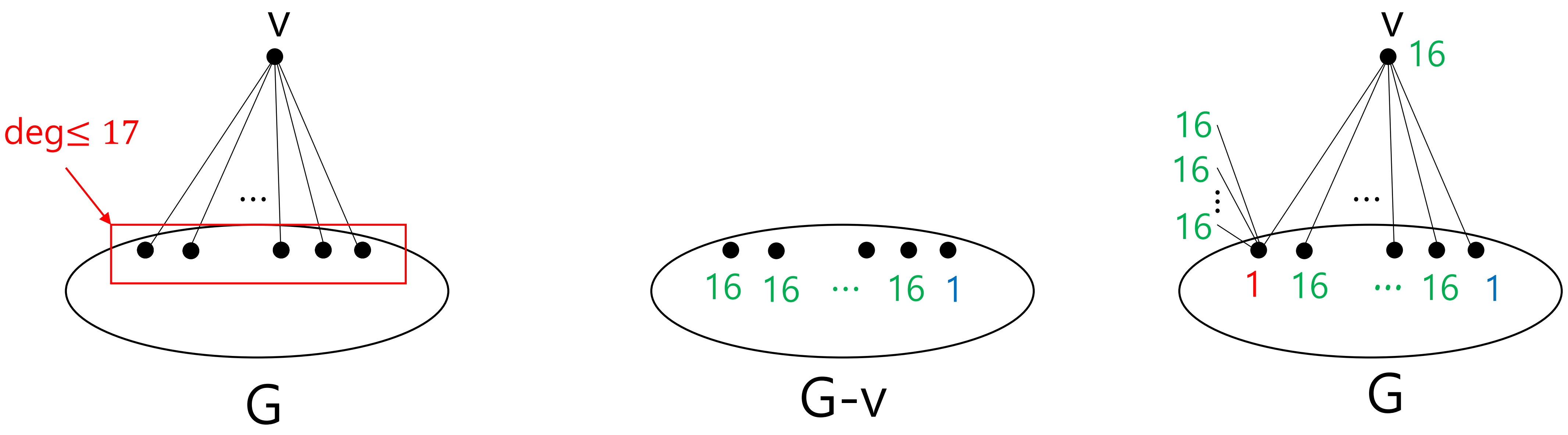

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Initial charge

degree	2	3	4	5	6	7	8
v=2d-6	-2	0	2	4	6	8	10
f=d-6	-4	-3	-2	-1	0	1	2

Initial charge sum = negative

Every planar graph with girth at least 5 is (1,16)-colorable.

degree	2	3	4	5	6	7	8
v=2d-6	-2	0	2	4	6	8	10
f=d-6	X	X	X	-1	0	1	2

Every planar graph with girth at least 5 is (1,16)-colorable.

degree	2	3	4	5	6	7	8
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Goal : every vertex has nonnegative final charge.
every face has nonnegative final charge.

Every planar graph with girth at least 5 is (1,16)-colorable.

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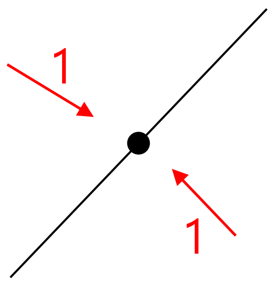
Discharging Rules

1. Each face sends charge 1 to every incident 2-vertex.



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Discharging Rules

1. Each face sends charge 1 to every incident 2-vertex.
2. Each vertex of degree ≥ 18 sends charge $\frac{5}{3}$ to every incident face.

Every planar graph with girth at least 5 is (1,16)-colorable.

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v=2d-6	-2	0	2	4	6	8	10
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Discharging Rules

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A vertex of degree 18 has initial charge $2 \times 18 - 6 = 30$.

$$\frac{30}{18} = \frac{5}{3}$$

Every planar graph with girth at least 5 is (1,16)-colorable.

degree	2	3	4	5	6	7	8
v=2d-6	-2	0	2	4	6	8	10
f=d-6	X	X	X	-1	0	1	2

Discharging Rules

1. Each face sends charge 1 to every incident 2-vertex.
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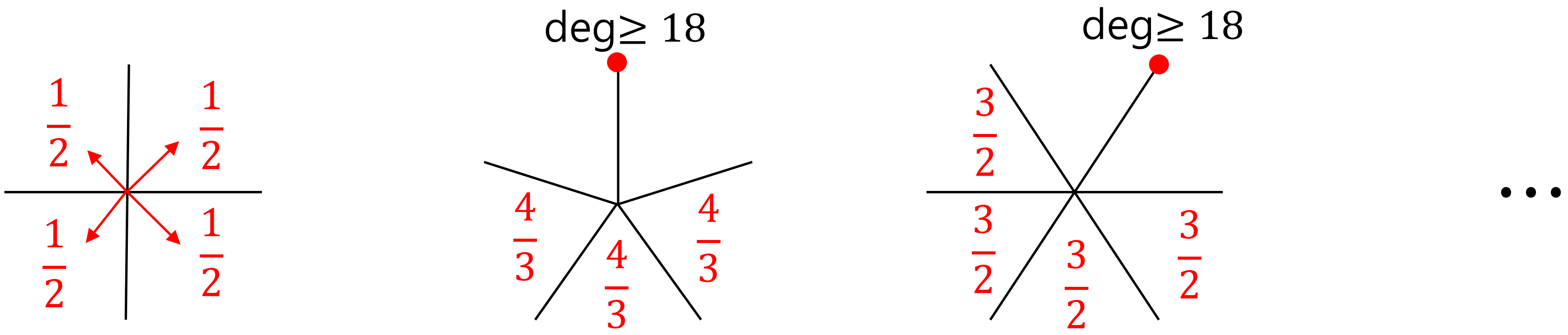
3. Each vertex of degree 4~17 distributes its initial charge as follows.



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1. Each face sends charge 1 to every incident 2-vertex.
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3. Each vertex of degree 4~17 distributes its initial charge as follows.

Then every vertex has nonnegative final charge.

Goal : every face has nonnegative final charge.

Every planar graph with girth at least 5 is (1,16)-colorable.

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v=2d-6	-2	0	2	4	6	8	10
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Case : face of length at least 6

Every planar graph with girth at least 5 is (1,16)-colorable.

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Case : face of length at least 6

Omit!!

Every planar graph with girth at least 5 is (1,16)-colorable.

Case : face of length 5 (it has initial charge -1)

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Case : face of length 5 (it has initial charge -1)

If f has many vertices of degree ≥ 18, then easy.

Every planar graph with girth at least 5 is (1,16)-colorable.

Case : face of length 5 (it has initial charge -1)

If f has many vertices of degree $\geq$ 18, then easy.
If f has no vertices of degree $\geq$ 18, then f has no 2-vertex. So, easy.
(By the second configuration)

Every planar graph with girth at least 5 is (1,16)-colorable.

Case : face of length 5 (it has initial charge -1)

If f has many vertices of degree ≥ 18, then easy.
If f has no vertices of degree ≥ 18, then f has no 2-vertex. So, easy.
(By the second configuration)

The worst case is when f has exactly one vertex of degree 18.

Every planar graph with girth at least 5 is (1,16)-colorable.

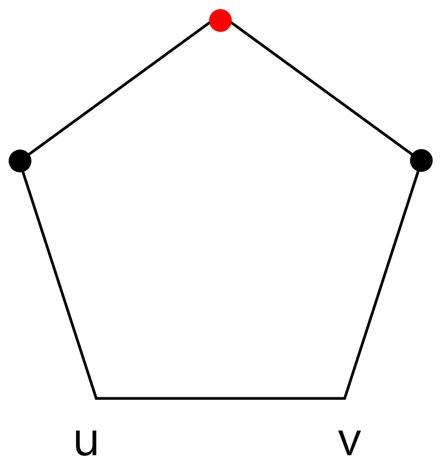
Case : face of length 5 (it has initial charge -1)

If f has many vertices of degree ≥ 18, then easy.
If f has no vertices of degree ≥ 18, then f has no 2-vertex. So, easy.
(By the second configuration)

The worst case is when f has exactly one vertex of degree 18
and f has two 2-vertices.

Every planar graph with girth at least 5 is (1,16)-colorable.

Assume f contains exactly one vertex of degree 18 and two 2-vertices.



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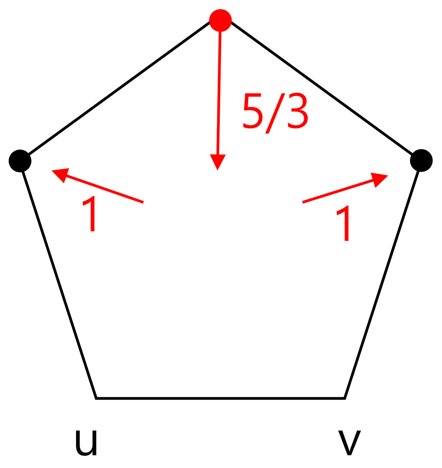
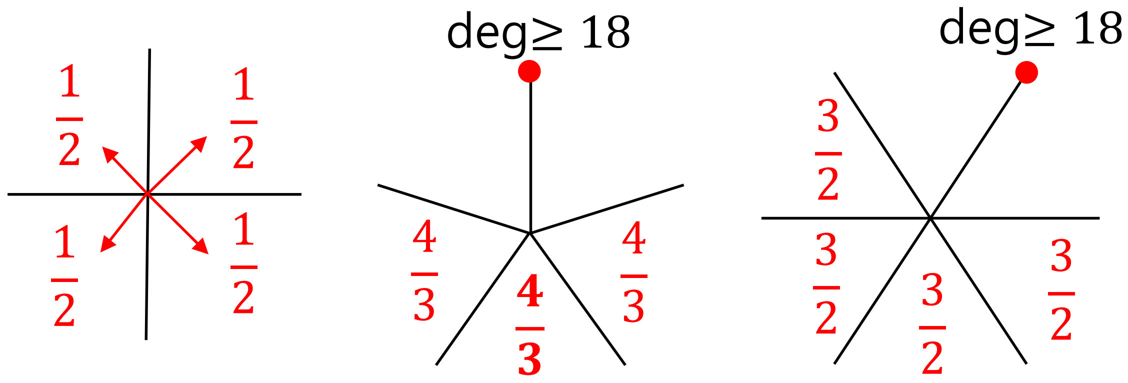


Recall that



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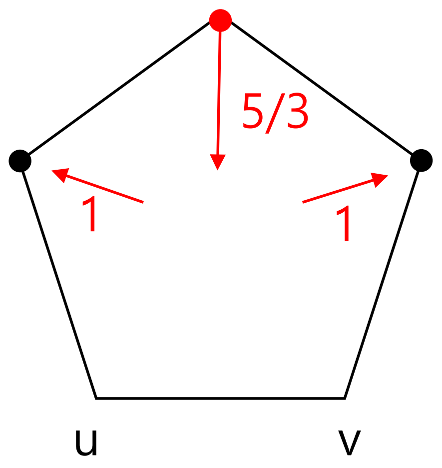

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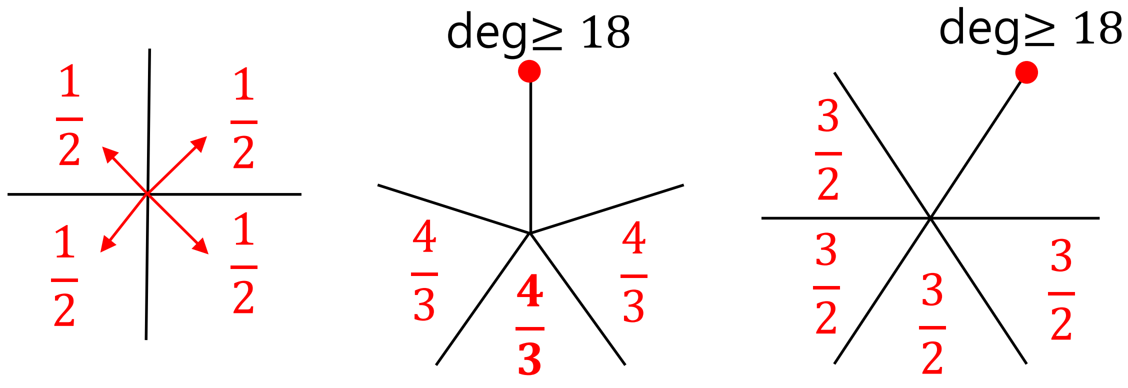

If one of u and v has degree at least 5,

$$\frac{5}{3}+\frac{4}{3}-1-1-1=0$$

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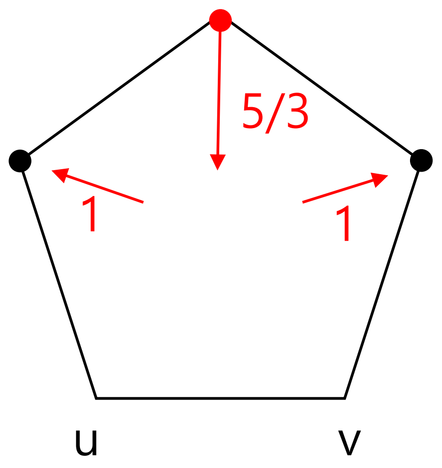

1) u=3, v=3
2) u=3, v=4
3) u=4, v=4

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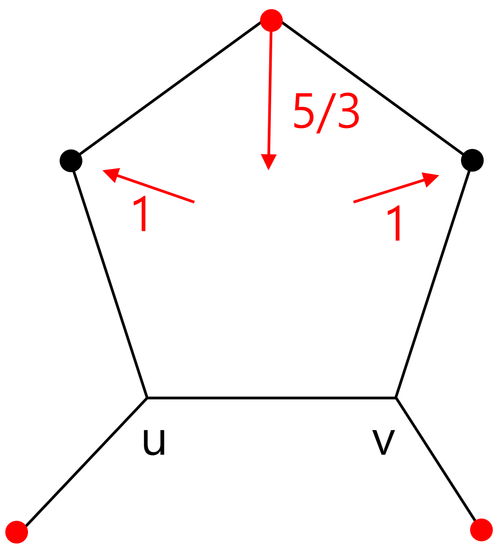
Assume f contains exactly one vertex of degree 18 and two 2-vertices.



1) u=3, v=3

the final charge is

$$\frac{5}{3} - 1 - 1 - 1 = -\frac{4}{3}$$

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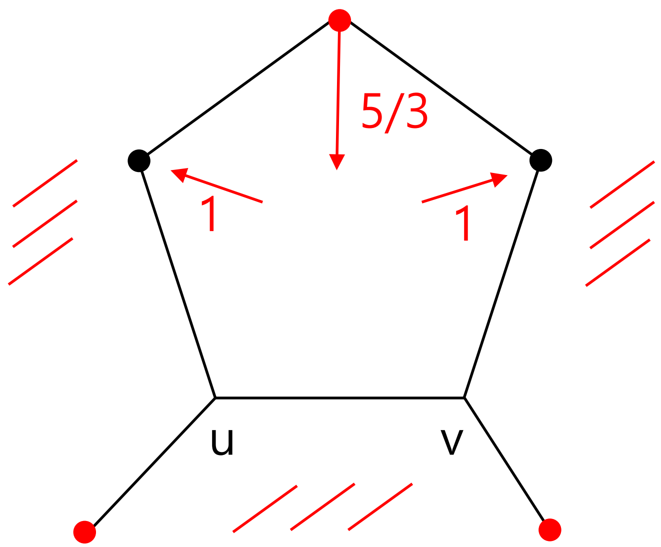

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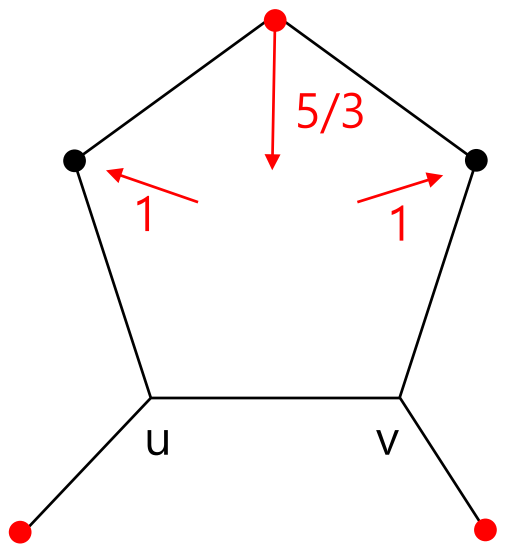
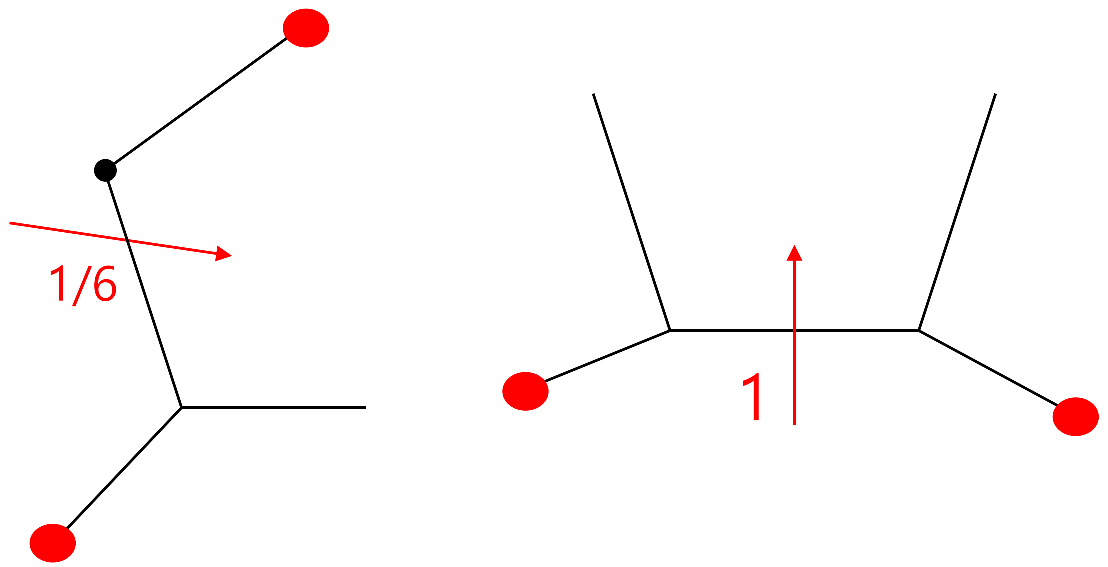
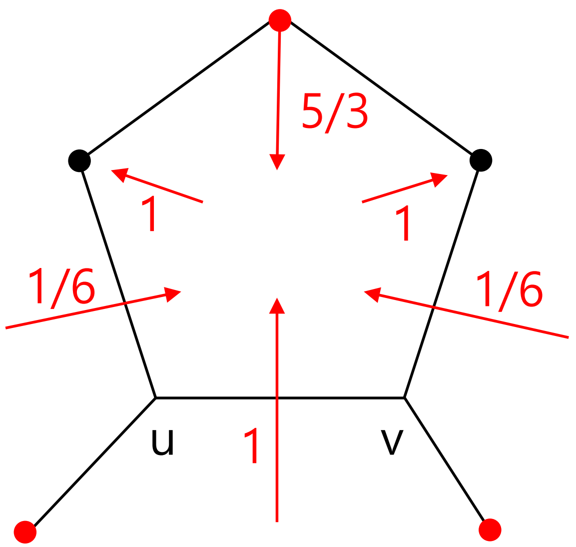




New Rules!!

Every planar graph with girth at least 5 is (1,16)-colorable.

Assume f contains exactly one vertex of degree 18 and two 2-vertices.



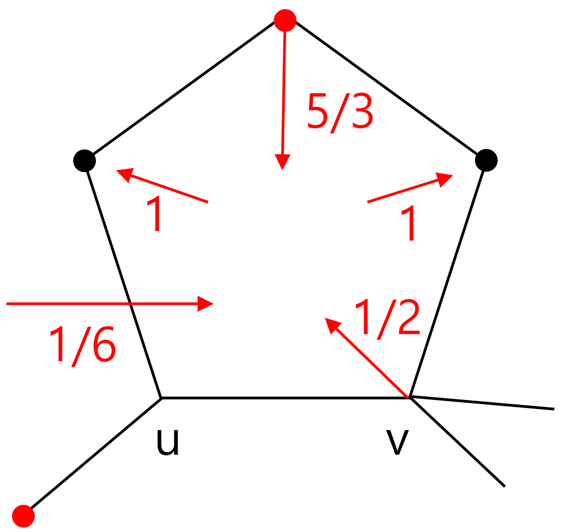
1) u=3, v=3

the final charge is

$$\frac{5}{3}+1+\frac{1}{6}+\frac{1}{6}-1-1-1=0$$

Every planar graph with girth at least 5 is (1,16)-colorable.

Assume f contains exactly one vertex of degree 18 and two 2-vertices.



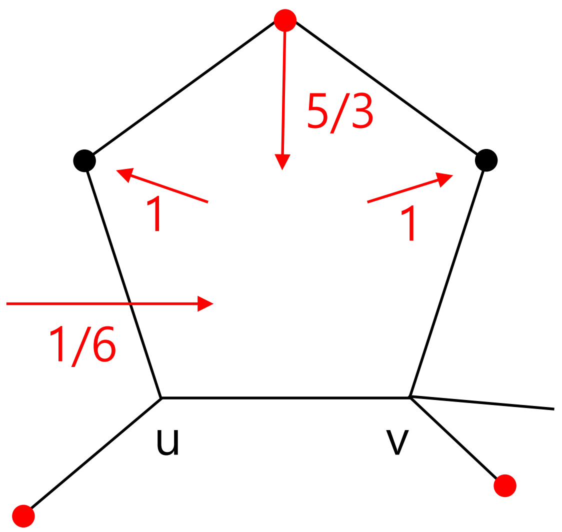
2) u=3, v=4

the final charge is

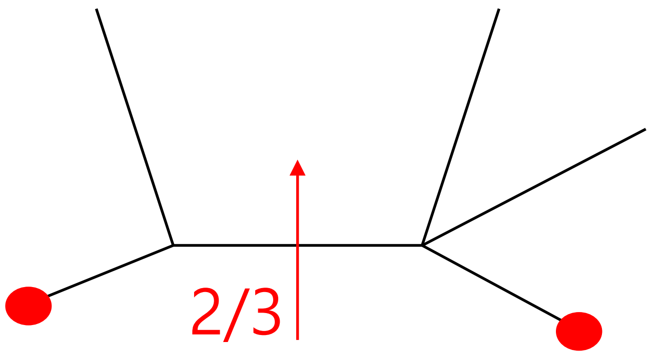
$$\frac{5}{3}+\frac{1}{2}+\frac{1}{6}-1-1-1=-\frac{2}{3}$$

Every planar graph with girth at least 5 is (1,16)-colorable.

Assume f contains exactly one vertex of degree 18 and two 2-vertices.



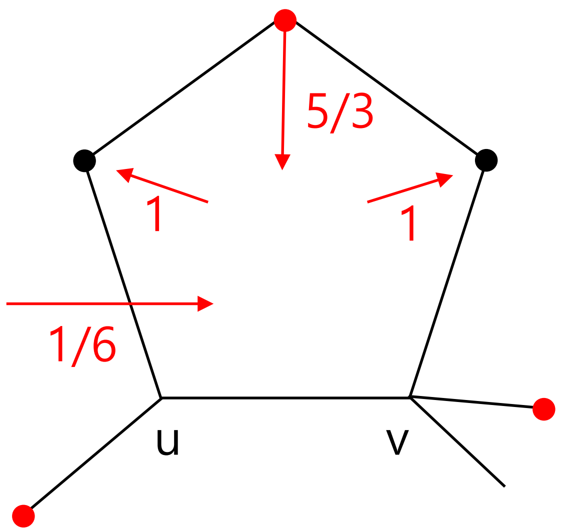
2) u=3, v=4



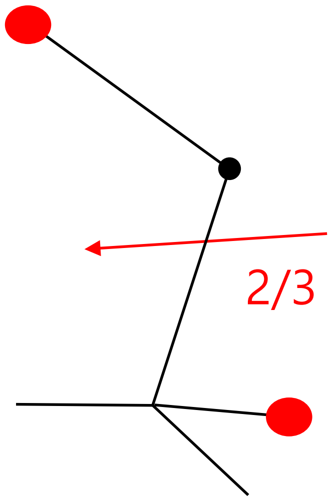
New Rule!!

Every planar graph with girth at least 5 is (1,16)-colorable.

Assume f contains exactly one vertex of degree 18 and two 2-vertices.



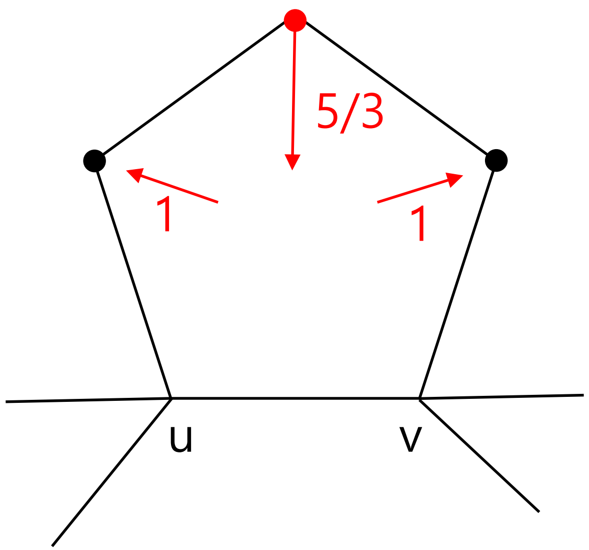
2) u=3, v=4



New Rule!!

Every planar graph with girth at least 5 is (1,16)-colorable.

Assume f contains exactly one vertex of degree 18 and two 2-vertices.



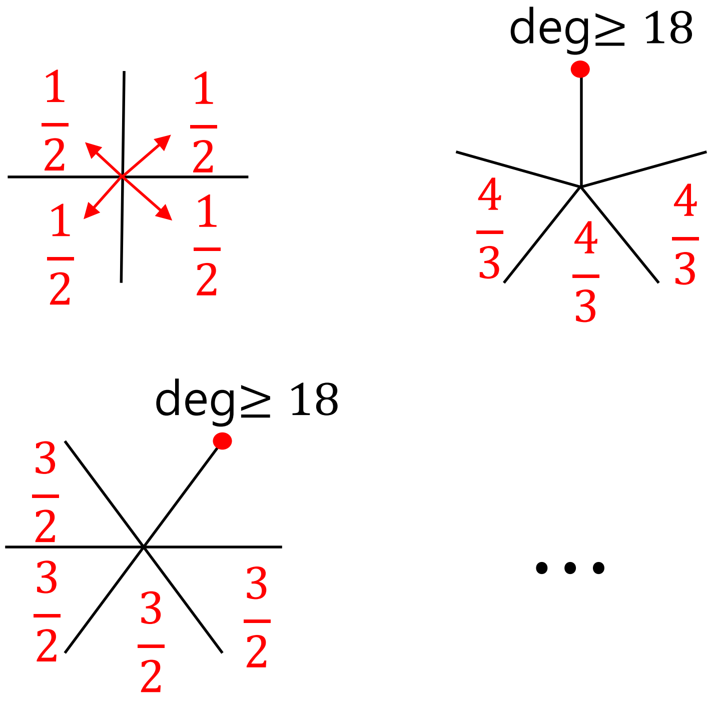
3) u=4, v=4

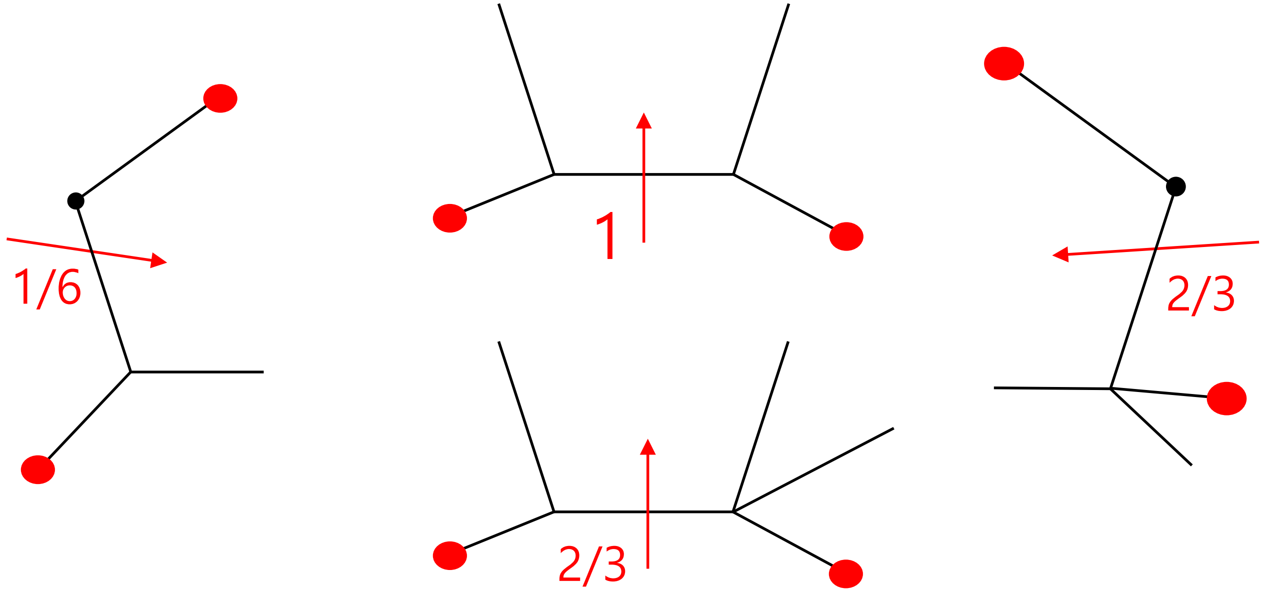
Similarly, we can do.

Every planar graph with girth at least 5 is (1,16)-colorable.

Repeat again with new rules.

1. Each face sends charge 1 to every incident 2-vertex.
2. Each vertex of degree ≥ 18 sends charge $\frac{5}{3}$ to every incident face.
3. Each vertex of degree 4~17 distributes its initial charge as follows.





Every planar graph with girth at least 5 is (1,16)-colorable.

Repeat again with new rules.

Then the final charge of each face is nonnegative.
So, the final charge sum is nonnegative.

There is no minimal counterexample.

Every planar graph with girth at least 5 is (1,16)-colorable.

How to prove (1,10)-colorable?

Find a new reducible configuration
and we change a rule so that
a vertex of degree ≥ 12 distributes its charge not uniformly.

Open problems

- Is every planar graph with girth at least 5 (1,9)-colorable?

  Note that every planar graph with girth at least 5 is (3,5)-colorable.

- Is every planar graph with girth at least 6 (1,3)-colorable?

- Is there a planar graph with girth at least 5 that is not (1,4)-colorable?

  Note that there is a planar graph with girth 5 that is not (1,3)-colorable.

Thank you

Consider a counterexample graph G

Find some configurations that cannot occur in G

there is no vertex of degree at most 3 (G is locally **dense**)

Assign some charges to the vertices and faces

in terms of degree

Move the charge around

The initial charge sum and the final charge sum are different

initial charge is negative since G is planar, which means **sparse**

So there is no counterexample

contradiction between locally dense and globally sparse

Thank you

Merit

1. Easy to start / learn

    One of my co-authors has not taken ‘Discrete math’.

Merit

2. Can do anywhere

In half time of the final match of the FIFA World Cup

Merit

2. Can do anywhere

Coffee break in ICM

Merit

2. Can do anywhere

Merit

1. Easy to start / learn
2. Can do anywhere
3. Many applications

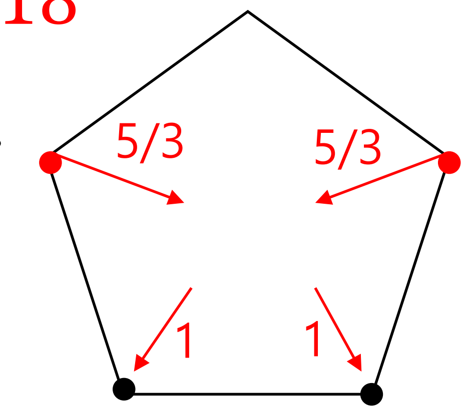
Every planar graph with girth at least 5 is (1,16)-colorable.

Case : face of degree(length) 5
(it has initial charge -1)

Note that a face loses its charge only if it has a vertex of degree 2.
So the maximum charge a face loses is 3.

1. Assume there exist at least two vertices of degree ≥ 18

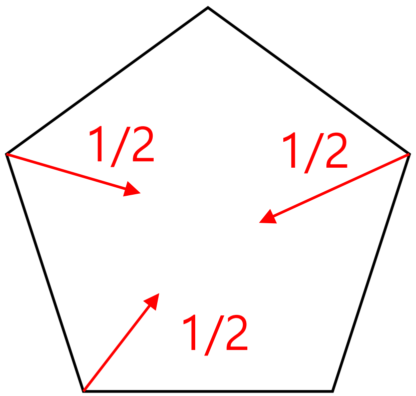
The final charge of the face $= \frac{5}{3} + \frac{5}{3} - 1 - 1 - 1 = \frac{1}{3} > 0.$



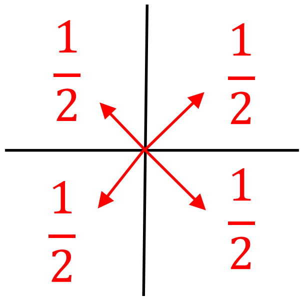
Every planar graph with girth at least 5 is (1,16)-colorable.

2. Assume there exists no vertex of degree ≥ 18
   (It means that there is no vertex of degree 2)

   If there are at least three vertices of degree ≥ 4,
   then the final charge is $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1=\frac{1}{2}>0$



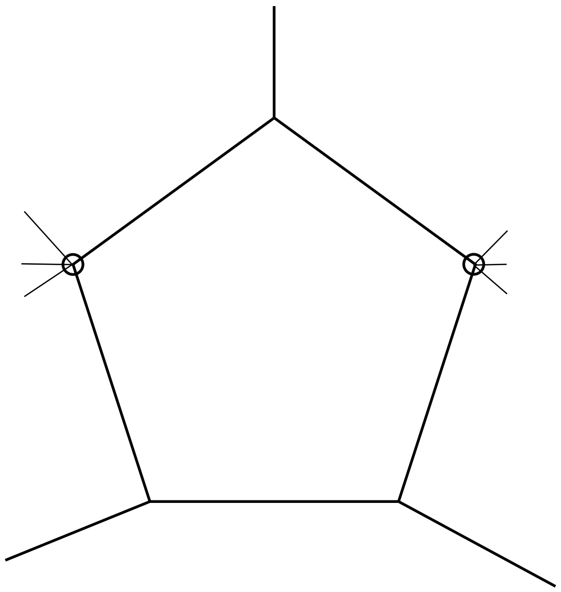
Recall that



Every planar graph with girth at least 5 is (1,16)-colorable.

2. Assume there exist no vertices of degree ≥ 18

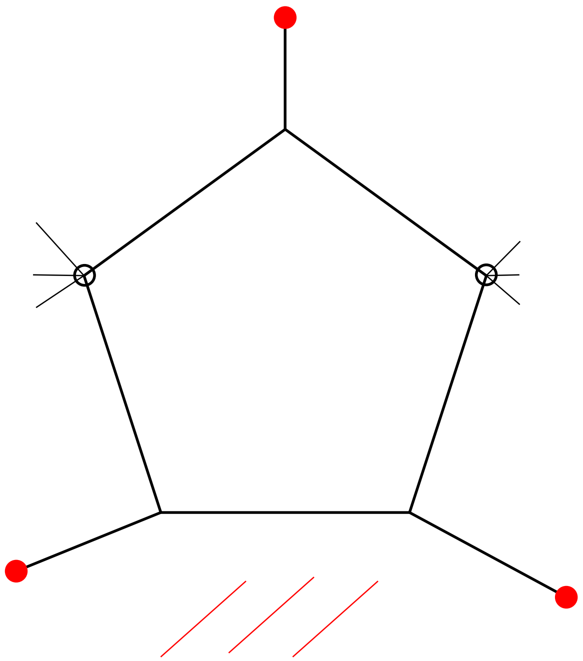
If there are at least three vertices of degree 3



Every planar graph with girth at least 5 is (1,16)-colorable.

2. Assume there exist no vertices of degree ≥ 18

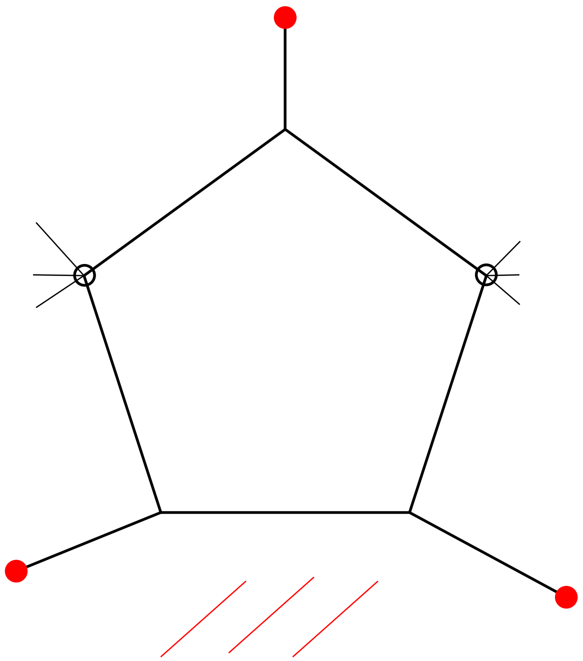
If there are at least three vertices of degree 3

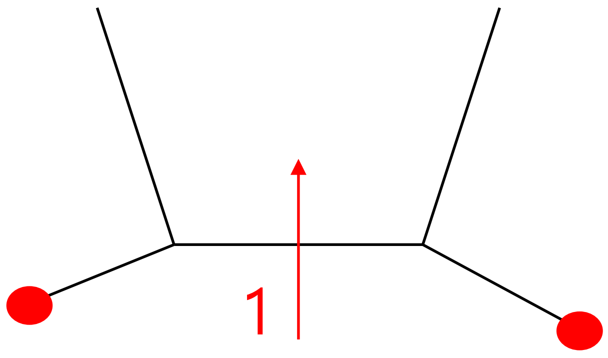


Every planar graph with girth at least 5 is (1,16)-colorable.

2. Assume there exist no vertices of degree ≥ 18

If there are at least three vertices of degree 3





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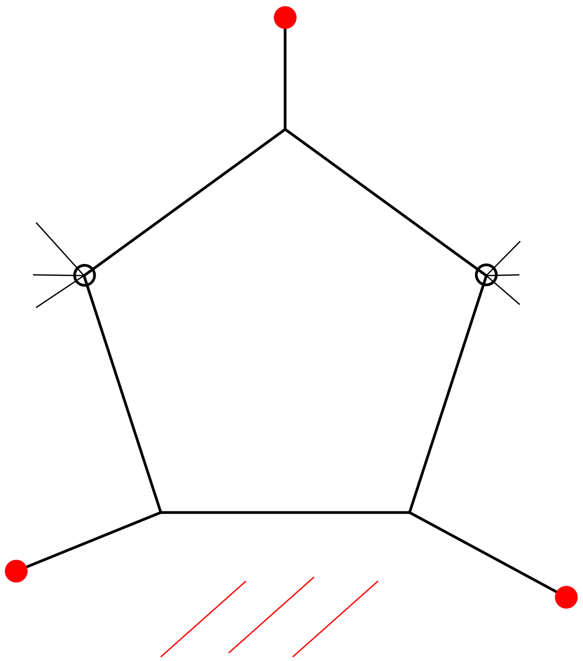
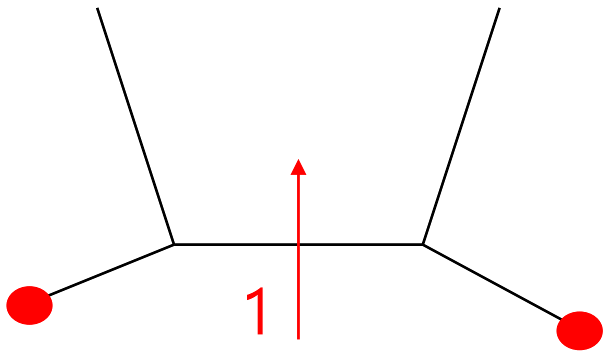


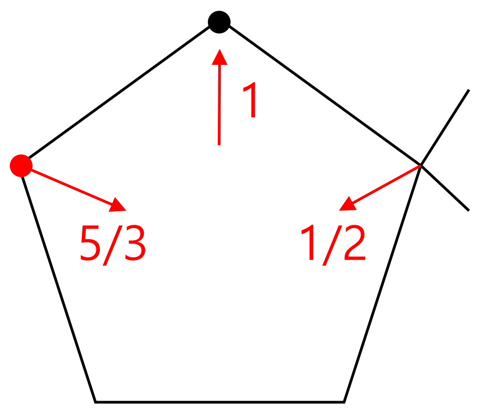


the final charge$=1-1+\varepsilon>0$

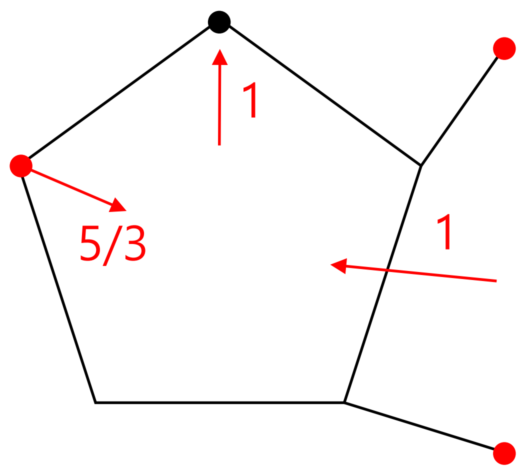
Every planar graph with girth at least 5 is (1,16)-colorable.

3. Assume there exists one vertex of degree ≥ 18

If there is at most one 2-vertex,



$$\frac{5}{3} + \frac{1}{2} - 1 - 1 = \frac{1}{6} > 0$$



$$\frac{5}{3} + 1 - 1 - 1 = \frac{2}{3} > 0$$